Values for team games

L. Hernández-Lamoneda and F. Sánchez-Sánchez

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1. Introduction

$N = \{1, \ldots, n\}$

Let k be a fix cardinality between 1 and $n-1$.

$B = \{S \subseteq N : |S| = k\}$ be the set of teams.

Definition 1. *By a team game we mean a function $v : B \to \mathbb{R}$. Let G_k be the set of team games.*

Problem. We want to divide an amount c in a "fair" way, among the players in a team game.

Definition 2. *By a solution on $G_k \times \mathbb{R}$ we mean a function*

$$\varphi : G_k \times \mathbb{R} \to \mathbb{R}^n.$$

Example 1. Let $k = 1$, then we could see the bankruptcy game $[c, x]$ as an element (x, c) of $G_1 \times \mathbb{R}$, where $v(\{i\}) = x_i$.

Example 2. A Market with m sellers and n buyers.
M set of sellers.
N set of buyers.
1 product.
Assumption: $m \leq n$

$$v(\{i, j\}) = \begin{cases} 1 & \text{if } i \in M \text{and } j \in N \\ 0 & \text{otherwise} \end{cases}$$

Example 3. Variation of the Assignment Game (Shapley and Shubik [1972]). Market in private homes.

m sellers.

n buyers.

c_i the value that seller gives to his house i.

h_{ij} the value that buyer j gives to the house i.

For a seller i and buyer j,

$$v(\{i,j\}) = \max\{0, h_{ij} - c_i\} = a_{ij}.$$

Example 4. When a traveler purchases a Europass he can choose to visit a subset of k countries (where $k = 3, 4$ or 5) from a set of N European countries, to travel for a certain fix amount of days d. We could think of a game (v, c) where $v \in G_k$ is such that $v(S) =$ the number of travelers (in a particular summer, say) that chose the set S of countries for his pass, and where c is the total amount collected, at the end of the period, from Europasses purchased.

Then a solution

$$\varphi : G_k \times \mathbb{R} \to \mathbb{R}^N$$

will assign to (v, c) the vector $\varphi(v, c)$ where $\varphi_j(v, c)$ is the amount that corresponds to country j.

Example 5. Simplification version of the game of poker.

$N = \{1, \ldots, 52\}$ set of cards.

S vs T with $|S| = |T| = 5$.

$v(S) =$ Expected value.

$\varphi_j(v, 0)$ value of card j.

2. General results

Proposition 1. *The space of linear, symmetric solutions*

$$\varphi : G_k \times \mathbb{R} \to \mathbb{R}^n$$

is 3-dimensional. Moreover, their general expression is given by

$$\varphi_i(v,c) = \alpha c + \beta \sum_{S \ni i} v(S) - \gamma \sum_{S \not\ni i} v(S) \tag{2.1}$$

for arbitrary $\alpha, \beta, \gamma \in \mathbb{R}$ *and* $i \in N$.

Definition 3. *(Efficiency Axiom) The solution* $\varphi : G_k \times \mathbb{R} \to \mathbb{R}^n$ *is said to be efficient if*

$$\varphi(v, c) \cdot \ 1_n = c.$$

Teorema 1. *The space of linear, symmetric solutions*

$$\varphi : G_k \times \mathbb{R} \to \mathbb{R}^n$$

that are also efficient is 1-dimensional. Their general expression is given by

$$\varphi_i(v, c) = \frac{c}{n} + \lambda \left[\sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n - s} \right]$$

for arbitrary $\lambda \in \mathbb{R}$.

2.1. **Interpretation.**

Step 1.

$\frac{c}{n} \rightarrow$ player i

Step 2.

For every S there is a transfer of $\lambda v(S)$ from de players in $N \backslash S$ to the players in S:

- Every $i \in N \backslash S$ pays $\frac{\lambda v(S)}{n-s}$.
- Every $i \in S$ receives $\frac{\lambda v(S)}{s}$.

For every $x \in \mathbb{R}^n$, let

$$x^k(S) = \sum_{i \in S} x_i$$

for every $S \in B$.

Definition 4. *(Naturalness Axiom) The solution* $\varphi : G_k \times \mathbb{R} \to \mathbb{R}^n$ *is said to be natural if*

$$\varphi(x^k, x \cdot 1_n) = x.$$

for every $x \in \mathbb{R}^n$.

Teorema 2. *There exists a unique solution* $\psi : G_k \times \mathbb{R} \to \mathbb{R}^n$ *which is linear, symmetric, efficient and natural. It is given by*

$$\psi_i(v,c) = \frac{c}{n} + \frac{n-1}{\binom{n}{s}} \left[\sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n-s} \right].$$

Remark *The formula for the unique linear, symmetric, efficient and natural solution,* ψ, *on* $G_k \oplus \mathbb{R}$ *takes the form:*

$$\psi(v,c) = \frac{c}{n} 1_n + (n-1)Sh(v).$$

Let us suppose that the amount to be share is equal to the sum of all the worth's coalitions, then it results reasonably that every player gets the proportional part of what he help to generate. So,

Axiom *(Gathering) We say that φ satisfies the gather axiom if*

$$\varphi_i(v, \sum_S v(S)) = \sum_{S \ni i} \frac{v(S)}{s}.$$

Teorema 3. *There exist a unique linear, symmetric, efficient and gather solution, φ, on $G_k \times \mathbb{R}$. Furthemore it takes the form:*

$$\varphi_i(v,c) = \frac{c}{n} + \frac{n-s}{n}\left[\sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n-s}\right].$$

3. Bankruptcy problem

- There is a 3-dimensional space of solutions $\varphi : G_1 \times \mathbb{R} \to \mathbb{R}^n$ for the bankruptcy problem. The general expression is given by

$$\varphi_i(x, c) = \alpha c + \beta x_i - \gamma \sum_{k \neq i} x_k$$

- There is a 1-dimensional space of linear symmetric and efficient solutions given by

$$\varphi_i(x, c) = \frac{c}{n} + \widetilde{\lambda}\,(x_i - \overline{x})$$

where $\overline{x}$ is the average of the coordinates of x.

- There exists a unique bankruptcy solution which is linear, symmetric, efficient and natural. It is given by

$$\varphi_i(x, c) = \frac{c}{n} + x_i - \overline{x}$$

4. Assignment Game

The solution of theorem 2 takes the following form,

$$\varphi_i(A) = \frac{c}{2n} + \sum_k \frac{a_{ki}}{2} - \sum_k \sum_{j \neq i} \frac{a_{kj}}{2n-2}$$

5. A Market with 1 sellers and 2 buyers

The solution is: $(1, 0, 0)$ the only imputation in the core.

6. Poker

j	$\varphi_j(v, 0)$
A	0.236088
K	0.128311
Q	0.066628
J	0.029374
10	0.005330
9	-0.012679
8	-0.027257
7	-0.040026
6	-0.051830
5	-0.063794
4	-0.076991
3	-0.090044
2	-0.103109

References

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